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A Solution to the Standard H^∞ Problem for Multivariable Distributed Systems*

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Abstract

In this note we summarize the results of our work [9] where we study the problem of the H^∞ optimal control of multivariable distributed systems in the four block setting. This is based on several previous papers and employs the skew Toeplitz framework developed in [1], [4], and [3].

1 Introduction

This is a summary of our work [9] where the standard H^∞-control problem for multivariable distributed systems is studied, in the four block setting, using skew Toeplitz theory originally developed in [1], [4] and [3].

It is well known that the four block H^∞ problem can be reduced to finding the singular values of a certain operator: so-called four block operator which is defined in [2] and [3]. The key point in this reduction, in the multivariable infinite dimensional version, is to identify the finite and infinite dimensional parts of the problem data. This point was shown in [6] for MIMO distributed stable plants and rational weights modeling the disturbances.

The main result of our paper [9] is that the singular values of the four block operator can be computed by an explicit rank type formula.

2 Problem Definition and Preliminary Remarks

In this note all Hardy spaces are defined on the unit disc D in the standard way. For an integer m we denote the canonical unilateral shift (defined by multiplication by z) on $H^2(\mathbf{C}^m)$ by $S : H^2(\mathbf{C}^m) \to H^2(\mathbf{C}^m)$ and the bilateral shift on $L^2(\mathbf{C}^m)$ by $U : L^2(\mathbf{C}^m) \to L^2(\mathbf{C}^m)$. Let W, F, G, J and M be H^∞ matrices, of sizes $p \times m$, $p \times l$, $q \times m$, $q \times l$ and $p \times p$ respectively, with $p \leq \max\{m, l\}$, where W, F, G, J have rational entries, and M is a nonconstant inner matrix. These matrices are associated with the weighting matrices and the plant in the usual way of transforming the standard problem to the 4-block framework (i.e. via Youla parametrization and some inner outer factorizations see e.g. [5]). It is important to note that for many problems of interest, in the case of rational weights and distributed stable plants, this reduces to the kind of problem described below. See [6] for all the details. The standard H^∞ problem reduces to finding

$$\mu := \inf\{\| \begin{bmatrix} W - MQ & F \\ G & J \end{bmatrix} \|_\infty \; : \; Q \in H^\infty \; p \times m\}, \qquad (1)$$

where for a $k \times n$ matrix of the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, ($A$, B, C, D having appropriate sizes with entries in L^∞) we set

$$\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \|_\infty = \text{ess sup}\{\| \begin{bmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{bmatrix} \| \; : |\zeta| = 1\} .$$

(For the norm on the right hand side the $k \times n$ matrix is taken as a linear operator from $\mathbf{C}^n$ to $\mathbf{C}^k$ for each fixed ζ in ∂D, the unit circle). Note that if $F = G = J = 0$ then this problem reduces to the classical Nehari problem, which is also known as the *one block problem*. For $F = J = 0$ we have the *two block problem*.

To the $p \times p$ inner matrix M, we associate the spaces $H(M) := H^2(\mathbf{C}^p) \ominus MH^2(\mathbf{C}^p)$ and $L(M) := L^2(\mathbf{C}^p) \ominus MH^2(\mathbf{C}^p)$. Let $P_{H(M)} : H^2(\mathbf{C}^p) \to H(M)$, $P_{L(M)} : L^2(\mathbf{C}^p) \to L(M)$, $P_{H^2} : L^2(\mathbf{C}^p) \to H^2(\mathbf{C}^p)$, and $P_{L^2 \ominus H^2} : L^2(\mathbf{C}^p) \to L^2(\mathbf{C}^p) \ominus H^2(\mathbf{C}^p)$ be orthogonal projections.

We now define the four block operator (see [2] and [3]):

$$A := \begin{bmatrix} P_{H(M)}W(S) & P_{L(M)}F(U) \\ G(S) & J(U) \end{bmatrix} .$$

Note that $A : H^2(\mathbf{C}^m) \oplus L^2(\mathbf{C}^l) \to L(M) \oplus L^2(\mathbf{C}^q)$.

In the paper, by a slight abuse of notation, ζ will denote a complex variable as well as an element of ∂D. The context will make the meaning clear. Note that $W(S)$ can be seen as the operator defined by multiplication by $W(\zeta)$, and similarly for $G(S)$, $F(U)$ and $J(U)$. Using the commutant lifting theorem [10], one can show that μ is equal to $\|A\|$. (See [2] and [3] for the details.) Note that $\|A\|^2$ is the largest element of $\sigma(A^*A)$, the spectrum of A^*A which consists of the discrete spectrum (i.e. eigenvalues with finite multiplicity) which we denote by $\sigma_d(A^*A)$, and its complement $\sigma_e(A^*A)$, the essential spectrum.

Note that when $\|A\|^2 \notin \sigma_e(A^*A)$, $\|A\|^2$ is an eigenvalue of A^*A. In [9] we developed a rank type formula for the eigenvalues of A^*A. This formula is obtained by a certain linear system of equations (called the *singular system*, [3]). These equations are derived from the inversion of two Toeplitz operators and the essential inversion of a skew Toeplitz operator. It is important to note that in the two block problem one of the Toeplitz operator inversions disappears, and in the one block case the same is true for both of the Toeplitz operator inversions. The Fredholm conditions on the invertibility of the skew Toeplitz operator (which is essentially invertible) and the coupling betweeen various systems of equations constitute the singular system. See also [1] and [3].

3 Summary of Main Results

3.1 Discrete Spectrum

Let us begin with the following notation, $W = B/k$, $F = C/k$, $G = D/k$, and $J = E/k$, where B, C, D, E are polynomial matrices

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and k is a scalar polynomial. We denote by n, an upper bound for the degree of the entries of all polynomial matrices appearing throughout the paper.

Now it is easy to see that ρ^2 is an eigenvalue of A^*A if and only if there exists a nonzero

$$\begin{bmatrix} x \\ y \end{bmatrix} \in H^2(\mathbf{C}^m) \oplus L^2(\mathbf{C}^l)$$

such that the following eigenvalue-eigenvector equation holds:

$$(\rho^2 k(S)^* k(S)\mathrm{I} - B(S)^* P_{H(M)} B(S) - D(S)^* D(S))x$$
$$-(P_{H^2}(B(U)^* P_{L(M)} C(U) + D(U)^* E(U)))y = 0 \ ,$$

and

$$-((C(U)^* P_{H(M)} B(S) + E(U)^* D(S)))x$$
$$+(\rho^2 k(U)^* k(U)\mathrm{I} - C(U)^* P_{L(M)} C(U) - E(U)^* E(U))y = 0 \ .$$

The key step in skew Toeplitz method is to compute $P_{H(M)}B(S)x$ and $P_{L(M)}C(U)y$ explicitly. In doing this, in the MIMO version of the problem ([1] and [9]) following factorizations are used:

$$M^*B = \Omega_b M_b^* \quad \text{and} \quad M^*C = \Omega_c M_c^*$$

where Ω_b, Ω_c are polynomial matrices and M_b, M_c are inner matrices. See [8] for a discussion on when these factorizations exist. One thing to note is that in the SISO distributed and in the MIMO finite dimensional cases these factorizations are trivial.

After further computations and factorizations of the above type we reduce the eigenvalue eigenvector equation to a set of equations from which we obtain finitely many interpolation conditions for ρ^2 to be an eigenvalue of A^*A. These conditions form a system of linear equations: *the singular system*. Our main result can be stated as follows.

Theorem: ρ^2 is an eigenvalue of A^*A if and only if a certain matrix, denoted by R_ρ, loses its rank at ρ. The matrix R_ρ as a function of ρ can be computed explicitly in terms of the problem data via the singular system.
Proof See [9].

The range of ρ in the above theorem can be taken from the essential norm of A (see next section) to an upper bound for μ which is easily obtained by just putting $Q = 0$ in (1). We search for the eigenvalues of A^*A by decreasing the values of ρ starting from this upper bound, and the first eigenvalue we find gives the H^∞ optimal performance μ. This way we have an explicitly computable finite rank type formula for μ.

3.2 Essential Spectrum

When $\|A\|^2$ is in the essential spectrum of A^*A then there is no first eigenvalue. But this means that the eigenvalues accumulate to $\|A\|^2$, so in this case we can find the norm up to a certain tolerance using above procedure. We would like to compute this value (which is the essential norm) before we start our procedure described above.

In this section we study the essential spectrum of A^*A which, by definition, consists of those $\lambda \in \mathbf{C}$ for which there exists

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} \in H^2(\mathbf{C}^m) \oplus L^2(\mathbf{C}^l) \ \text{ with } \ \|\begin{bmatrix} x_n \\ y_n \end{bmatrix}\|_2 = 1 \ \ \forall\, n \geq 1$$

and $\begin{bmatrix} x_n \\ y_n \end{bmatrix} \to 0$ weakly as $n \to \infty$, such that

$$(\lambda\mathrm{I} - A^*A)\begin{bmatrix} x_n \\ y_n \end{bmatrix} \to 0 \text{ as } n \to \infty.$$

The essential norm, denoted by $\|A\|_e$, is defined as

$$\|A\|_e^2 = \max\{\lambda \ : \ \lambda \in \sigma_e(A^*A)\}.$$

In the SISO case we have that (see [3], Theorem 3.2)

$$\|A\|_e = \max(\alpha, \beta, \gamma)$$

where

$$\alpha = \max\{\|\begin{bmatrix} W(\zeta) & F(\zeta) \\ G(\zeta) & J(\zeta) \end{bmatrix}\| \ : \ \zeta \in \sigma_e(T)\},$$
$$\beta = \max\{\|[G(\zeta) \quad J(\zeta)]\| \ : \ \zeta \in \partial D\},$$
$$\gamma = \max\{\|\begin{bmatrix} F(\zeta) \\ J(\zeta) \end{bmatrix}\| \ : \ \zeta \in \partial D\} \ .$$

$\sigma_e(T)$ denotes the essential spectrum of the operator

$$T := P_{H(M)} S|_{H(M)}$$

We let $\mathcal{R}$ be the set of all $\lambda \in \partial D$ which do not lie on any of the open arcs of ∂D on which $M(\zeta)$ is a unitary operator-valued analytic function. Then from [7] and [10], we have that $\sigma_e(T) = \mathcal{R}$. In the case of infinite dimensional MIMO systems it may be difficult to find the essential norm of A. Nevertheless, upper and lower bounds can be established ([9]) in terms of α, β, γ. This is obtained under an assumption of the form: the Toeplitz operator with symbol $M_1^* M_0$ is invertible, where M_1, M_0 are some inner matrices (related to each other), determined from the problem data; see [9] for the precise definitions of these matrices. Under this assumption we can prove the following.

Proposition 1: Let $\alpha \leq \max\{\beta, \gamma\}$ then
(i) If $\gamma \geq \beta$ then $\|A\|_e = \gamma$.
(ii) If $\gamma < \beta$ then $\gamma \leq \|A\|_e \leq \beta$.

Proposition 2: In the finite dimensional MIMO case, i.e. M is rational, we have $\|A\|_e = \max\{\beta, \gamma\}$.

Proofs: See [9].

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